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Equivalences on Acyclic Orientations

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Abstract The cyclic and dihedral groups can be made to act on the set $\mathsf{Acyc}(Y)$ of acyclic orientations of an undirected graph Y, and this gives rise to the equivalence relations $\sim_\kappa$ and $\sim_\delta$, respectively. These two actions and their corresponding equivalence classes are closely related to combinatorial problems arising in the context of Coxeter groups, sequential dynamical systems, the chip-firing game, and representations of quivers.

In this paper we construct the graphs $C(Y)$ and $D(Y)$ with vertex sets $\mathsf{Acyc}(Y)$ and whose connected components encode the equivalence classes. The number of connected components of these graphs are denoted $\kappa(Y)$ and $\delta(Y)$, respectively. We characterize the structure of $C(Y)$ and $D(Y)$, show how $\delta(Y)$ can be derived from $\kappa(Y)$, and give enumeration results for $\kappa(Y)$. Moreover, we show how to associate a poset structure to each κ-equivalence class, and we characterize these posets. This allows us to create a bijection from $\mathsf{Acyc}(Y)/\sim_\kappa$ to $\mathsf{Acyc}(Y')/\sim_\kappa \cup \mathsf{Acyc}(Y'')/\sim_\kappa$, where Y' and Y'' denote edge deletion and edge contraction for a cycle-edge in Y, respectively, which in turn shows that $\kappa(Y)$ may be obtained by an evaluation of the Tutte polynomial at $(1, 0)$.

1 Introduction

An acyclic orientation O_Y of an undirected graph Y induces a partial ordering on the vertex set $\mathrm{v}[Y]$ by $i \leq_{O_Y} j$ if there is a directed path from i to j

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in O_Y. A cyclic 1-shift (left) of a linear extension of O_Y corresponds to converting a source of O_Y into a sink. This source-to-sink operation on $\mathsf{Acyc}(Y)$ gives rise to the equivalence relation denoted $\sim_\kappa$. Reversing a linear extension of O_Y corresponds to reflecting all edge orientations in O_Y. The coarser equivalence relation on $\mathsf{Acyc}(Y)$ obtained through source-to-sink operations and reflections is denoted $\sim_\delta$.

This paper is organized as follows. In Section 3 we construct the equivalence relations $\sim_\kappa$ and $\sim_\delta$, and graphs $C(Y)$ and $D(Y)$ that have vertex set $\mathsf{Acyc}(Y)$, and whose connected components corresponds to κ- and δ-equivalence classes, respectively. We let $\kappa(Y)$ and $\delta(Y)$ denote the number of equivalence classes. In Section 4 we study the structure and properties of the graphs $C(Y)$ and $D(Y)$, show how $\delta(Y)$ can be determined from $\kappa(Y)$, and give bounds for these quantities. In Section 5 we show that one may associate a poset to each κ-equivalence class and use this to establish the bijection

$$\varTheta\colon \mathsf{Acyc}(Y)/{\sim_\kappa} \longrightarrow \big(\mathsf{Acyc}(Y'_e)/{\sim_\kappa}\big) \bigcup \big(\mathsf{Acyc}(Y''_e)/{\sim_\kappa}\big) \;, \tag{1}$$

where Y'_e and Y''_e are the graphs formed by deleting and contracting a non-bridge edge e of Y, respectively. This leads to a new proof of a recursion relation for $\kappa(Y)$ in [9]. Finally, in the summary, we discuss how these constructions arise in other areas of mathematics, such as sequential dynamical systems, Coxeter groups, the chip-firing game, and the representation theory of quivers.

2 Terminology and Background

Let Y be an undirected, simple and loop-free graph with vertex set $\mathrm{v}[Y] = \{1, 2, \ldots, n\}$ and edge set $\mathrm{e}[Y]$. We let S_Y denote the set of total orders (i.e., permutations) of $\mathrm{v}[Y]$. In [13] an equivalence relation $\sim_Y$ is introduced on S_Y through the *update graph* $U(Y)$ of Y. The update graph has vertex set S_Y, and two distinct vertices $\pi = (\pi_i)_i$ and $\pi' = (\pi'_i)_i$ are adjacent if they differ in exactly two consecutive elements π_k and π_{k+1} such that $\{\pi_k, \pi_{k+1}\} \notin \mathrm{e}[Y]$. The equivalence relation $\sim_Y$ is defined by $\pi \sim_Y \pi'$ if π and π' are connected in $U(Y)$. We denote the equivalence class containing π by $[\pi]_Y$, and set

$$S_Y/{\sim_Y} = \big\{[\pi]_Y \mid \pi \in S_Y\big\} \;.$$

This corresponds to partially commutative monoids as defined in [3], but restricted to fixed length permutations over $\mathrm{v}[Y]$ and with commutation relations encoded by non-adjacency in the graph Y.

Orientations of Y are represented as maps $O_Y\colon \mathrm{e}[Y] \longrightarrow \mathrm{v}[Y] \times \mathrm{v}[Y]$, which may also be viewed as directed graphs. The set of acyclic orientations of Y is denoted $\mathsf{Acyc}(Y)$, and we set $\alpha(Y) = |\mathsf{Acyc}(Y)|$. Every acyclic orientation defines a partial ordering on $\mathrm{v}[Y]$ where the covering relations are $i \leq_{O_Y} j$ if $\{i, j\} \in \mathrm{e}[Y]$ and $O_Y(\{i, j\}) = (i, j)$. The set of linear extensions of O_Y contains precisely the permutations $\pi \in S_Y$ such that if $i \leq_{O_Y} j$, then i precedes j in π. Through the ordering of $\mathrm{v}[Y]$, every permutation $\pi \in S_Y$ induces a canonical linear order on $\mathrm{v}[Y]$, see [13]. Moreover, each permutation $\pi \in S_Y$ induces an acyclic orientation $O_Y^\pi \in \mathsf{Acyc}(Y)$ defined by $O_Y^\pi(\{i,j\}) = (i,j)$ if i precedes j in π and $O_Y^\pi(\{i,j\}) = (j,i)$ otherwise. The bijection

$$f_Y \colon S_Y/{\sim_Y} \longrightarrow \mathsf{Acyc}(Y) \;, \qquad f_Y([\pi]_Y) = O_Y^\pi \;, \tag{2}$$

from [13] allows us to identify equivalence classes and acyclic orientations. The number of equivalence classes under $\sim_Y$ is therefore given by $\alpha(Y)$.

For $O_Y \in \mathsf{Acyc}(Y)$ and $e = \{v,w\} \in \mathrm{e}[Y]$ let $O_Y^{\rho(e)}$ be the orientation of Y obtained from O_Y by reversing the edge-orientation of e. Let Y'_e and Y''_e denote the graphs obtained from Y by deletion and contraction of e, respectively, and let $O_{Y'}$ and $O_{Y''}$ denote the induced orientations of O_Y under these operations. The bijection

$$\beta_e \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}(Y'_e) \cup \mathsf{Acyc}(Y''_e) \tag{3}$$

defined by

$$O_Y \longmapsto \begin{cases} O'_Y \;, & O_Y^{\rho(e)} \notin \mathsf{Acyc}(Y), \\ O'_Y \;, & O_Y^{\rho(e)} \in \mathsf{Acyc}(Y) \text{ and } O_Y(e) = (v,w) \;, \\ O''_Y \;, & O_Y^{\rho(e)} \in \mathsf{Acyc}(Y) \text{ and } O_Y(e) = (w,v) \;. \end{cases}$$

is well-known, and shows that one may compute $\alpha(Y)$ through the recursion relation

$$\alpha(Y) = \alpha(Y'_e) + \alpha(Y''_e) \;,$$

valid for any $e \in \mathrm{e}[Y]$.

3 Graph Constructions for Equivalence Relations

3.1 Relations on $S_Y/{\sim_Y}$

Using cycle notation, let $\sigma, \rho \in S_n$ be the elements

$$\sigma = (n, n-1, \dots, 2, 1) \;, \qquad \rho = (1,n)(2,n-1)\cdots(\lceil \tfrac{n}{2} \rceil, \lfloor \tfrac{n}{2} \rfloor + 1) \;,$$

and let C_n and D_n be the subgroups

$$C_n = \langle \sigma \rangle \quad \text{and} \quad D_n = \langle \sigma, \rho \rangle \;. \tag{4}$$

Both C_n and D_n act on S_Y via $g \cdot (\pi_1, \dots, \pi_n) = (\pi_{g^{-1}(1)}, \dots, \pi_{g^{-1}(n)})$. Define $\boldsymbol{\sigma}_s(\pi) = \sigma^s \cdot \pi$, so that, e.g. $\boldsymbol{\sigma}_1(\pi) = \sigma \cdot \pi = (\pi_2, \pi_3, \dots, \pi_n, \pi_1)$, and define $\boldsymbol{\rho}(\pi) = \rho \cdot \pi = (\pi_n, \pi_{n-1}, \dots, \pi_2, \pi_1)$. We construct two undirected graphs $C(Y)$ and $D(Y)$ whose vertex sets are $S_Y/{\sim_Y}$, and edge sets are

$$\begin{aligned} \mathrm{e}[C(Y)] &= \Big\{ \{[\pi]_Y, [\boldsymbol{\sigma}_1(\pi)]_Y\} \mid \pi \in S_Y \Big\} \;, \\ \mathrm{e}[D(Y)] &= \Big\{ \{[\pi]_Y,\, [\boldsymbol{\rho}(\pi)]_Y\} \mid \pi \in S_Y \Big\} \cup \mathrm{e}[C(Y)] \;. \end{aligned}$$

Define $\kappa(Y)$ and $\delta(Y)$ to be the number of connected components of $C(Y)$ and $D(Y)$, respectively. By construction, $C(Y)$ is a subgraph of $D(Y)$ and $\delta(Y) \leq \kappa(Y)$.

Example 1 Let Y be the complete bipartite graph $K_{2,3}$, where the partition of the vertex set is $\{\{1,3,5\},\{2,4\}\}$. The graph $U(K_{2,3})$ is shown in Figure 1 with vertex labels omitted. By simply counting the components we see that $\alpha(K_{2,3}) = 46$. We can better understand the component structure of

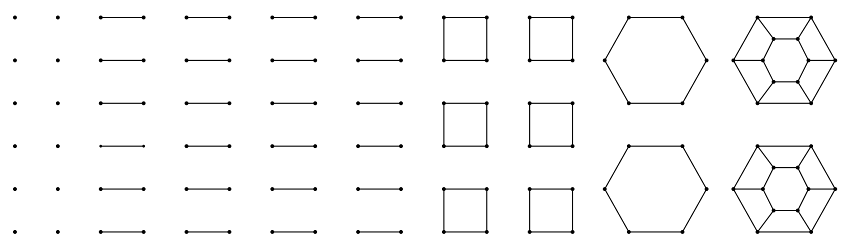

Fig. 1 The update graph $U(K_{2,3})$.

$U(K_{2,3})$ by mapping permutations as $(\pi_i)_i \overset{\phi}{\mapsto} (\pi_i \mod 2)_i$. Non-adjacency in Y coincides with parity, that is, if $\pi \sim_Y \sigma$, then $\phi(\pi) = \phi(\sigma)$. Through the map ϕ we see that the 12 singleton points in $U(K_{2,3})$ are precisely those with image 10101. Each of the 24 size-two components correspond to a pair of permutations with ϕ-image of the form 01011, 11010, 01101, or 10110. The six square-components arise from the permutations with ϕ-image 10011 and 11001. Finally, the permutations in the two hexagon-components are of the form 01110, and those in the two largest components have ϕ-image of the form 11100 or 00111.

The graphs $C(K_{2,3})$ and $D(K_{2,3})$ are shown in Figure 2. The dashed lines are edges that belong to $D(K_{2,3})$ but not to $C(K_{2,3})$. The vertices

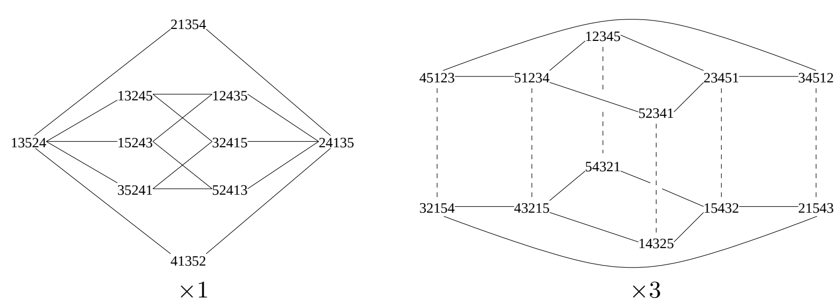

Fig. 2 The graph $C(K_{2,3})$ contains the component on the left, and three isomorphic copies of the structure on the right (but with different vertex labels). The dashed lines are edges in $D(K_{2,3})$ but not in $C(K_{2,3})$.

in Figure 2 are labeled by a permutation in the corresponding equivalence class in $S_Y/\sim_Y$. There are three isomorphic copies of the component on the right, but only one is shown. Each of these three components contains permutations whose ϕ-image is in $\{01101, 11010, 10101, 01011, 10110\}$. The component on the left contains all of the remaining permutations, i.e., all π for which $\phi(\pi) \in \{11100, 11001, 10011, 00111, 01110\}$. Clearly, $\kappa(K_{2,3}) = 7$ and $\delta(K_{2,3}) = 4$.

Permutations from $\sim_Y$ classes belonging to the same component in $C(Y)$ are called *κ-equivalent* permutations, as are the corresponding acyclic orientations. For two κ-equivalent permutations π and π' there is a sequence of adjacent non-edge transpositions and cyclic shifts that map π to π'. This is simply a consequence of the definitions of $S_Y/\sim_Y$ and $C(Y)$. Similarly, two permutations belonging to $\sim_Y$ classes on the same connected component in $D(Y)$ are called *δ-equivalent*, as are their corresponding acyclic orientations.

3.2 Relations on $\mathsf{Acyc}(Y)$

The bijection between $S_Y/\sim_Y$ and $\mathsf{Acyc}(Y)$ in (2) allows us to identify $[\pi]_Y$ with the acyclic orientation O_Y^π. It is clear that mapping $\pi \in [\pi']_Y$ to $\boldsymbol{\sigma}_1(\pi)$ corresponds precisely to converting the vertex π_1 from a source to a sink in O_Y^π. This can be extended. Following [15] we call the conversion of a source vertex to a sink vertex in $O_Y \in \mathsf{Acyc}(Y)$ a *source-to-sink operation*, or a *click*. Two orientations $O_Y, O_Y' \in \mathsf{Acyc}(Y)$ where O_Y can be transformed into O_Y' by a sequence of clicks are said to be click-related, and we write this as $\mathbf{c}(O_Y) = O_Y'$ where $\mathbf{c} = c_1 c_2 \cdots c_k$ with $c_i \in \mathrm{v}[Y]$. It is straightforward to verify that this click-relation is an equivalence relation on $\mathsf{Acyc}(Y)$. Through the bijection in (2) it is clear that a source-to-sink operation precisely encodes adjacency in the graph $C(Y)$, and the number of click-equivalence classes in $\mathsf{Acyc}(Y)$ therefore equals $\kappa(Y)$. The second equivalence relation on $\mathsf{Acyc}(Y)$ arises in the same manner by additionally identifying O_Y^π and the reverse orientation $O_Y^{\boldsymbol{\rho}(\pi)}$, the unique orientation that satisfies $O_Y^\pi(\{i,j\}) \neq O_Y^{\boldsymbol{\rho}(\pi)}(\{i,j\})$ for every $\{i,j\} \in \mathrm{e}[Y]$.

4 Structure of $C(Y)$ and $D(Y)$

The following result gives insight into the component structure of the graph $C(Y)$.

Proposition 1 *Let Y be a connected graph on n vertices and let $g, g' \in C_n$ with $g \neq g'$. Then $[g \cdot \pi]_Y \neq [g' \cdot \pi]_Y$.*

Proof Assume $g \neq g'$ with $[g \cdot \pi]_Y = [g' \cdot \pi]_Y$. By construction, we have $g \cdot \pi = \boldsymbol{\sigma}_s(\pi)$ and $g' \cdot \pi = \boldsymbol{\sigma}_{s'}(\pi)$. Without loss of generality we may assume $s' < s$. Let $V' \subset V = \mathrm{v}[Y]$ be the initial subsequence of vertices in $\boldsymbol{\sigma}_{s'}(\pi)$ that occurs at the end in $\boldsymbol{\sigma}_s(\pi)$. If any of the vertices in V' are adjacent to any of the vertices in $V \setminus V'$ in Y it would imply that

$[\boldsymbol{\sigma}_s(\pi)]_Y \neq [\boldsymbol{\sigma}_{s'}(\pi)]_Y$. The only possibility is that Y is not connected, but this contradicts the assumptions of the proposition. $\square$

There is a similar result to Proposition 1 for D_n, albeit somewhat more restrictive.

Proposition 2 *Let Y be a connected graph on n vertices and let $g, g' \in D_n$ with $g \neq g'$. If $[g \cdot \pi]_Y = [g' \cdot \pi]_Y$ holds then Y must be bipartite.*

Proof From $[g \cdot \pi]_Y = [g' \cdot \pi]_Y$ it follows from Proposition 1 that g and g' lie in different cosets of C_n in D_n. Without loss of generality we may assume that $g = \sigma^s$ and $g' = \rho\sigma^{s'}$. Let $m = |s' - s|$ and $m' = n - m$. If $s' > s$ (resp. $s' < s$) the first (resp. last) m elements of $g \cdot \pi$ and $g' \cdot \pi$ are the same but occur in reverse order. Call the set of these elements V_1. The remaining m' elements occur in reverse order as well in the two permutations. Let V_2 denote the set of these elements. For $[g \cdot \pi]_Y = [g' \cdot \pi]_Y$ to hold, there cannot be an edge between any two vertices in V_1, or between any two vertices in V_2. Therefore, the graph Y must be a subgraph of $K(V_1, V_2)$, the complete bipartite graph with vertex sets V_1 and V_2. $\square$

Remark 1 The pairs $(\sigma^s, \rho\sigma^{s'})$ and $(\sigma^{s'}, \rho\sigma^s)$ determine the same bipartite graph in the proof above. Also, the vertex sets V_1 and V_2 can only consist of consecutive elements in π.

Remark 2 If Y is connected and bipartite then $|\{[g \cdot \pi]_Y \mid g \in D_n\}| = 2n-1$. This follows from the fact that at most two $\sim_Y$ classes can coincide as all distinct pairs g and g' for which equality holds leads to different sets $V_1(\{g, g'\})$ and $V_2(\{g, g'\})$ modulo Remark 1. The existence of two or more distinct partitions of $\mathrm{v}[Y]$ into sets V_1 and V_2 as above would imply that Y is not connected.

We next consider the quantity $\delta(Y)$, and will show how it is determined by $\kappa(Y)$.

Lemma 1 *The reflection map $\boldsymbol{\rho}\colon S_Y \longrightarrow S_Y$ extends to an involution*

$$\boldsymbol{\rho}^*\colon \mathsf{Acyc}(Y)/\!\sim_\kappa \longrightarrow \mathsf{Acyc}(Y)/\!\sim_\kappa \ . \tag{5}$$

Proof By the definition of $U(Y)$ it follows that if $\pi, \pi' \in S_Y$ are adjacent in $U(Y)$ then so are $\boldsymbol{\rho}(\pi)$ and $\boldsymbol{\rho}(\pi')$. It follows easily that $\pi \sim_Y \pi'$ implies $\boldsymbol{\rho}(\pi) \sim_Y \boldsymbol{\rho}(\pi')$. The map $\boldsymbol{\rho}$ therefore extends to a map $\hat{\boldsymbol{\rho}}\colon S_Y/\!\sim_Y \longrightarrow S_Y/\!\sim_Y$ by $\boldsymbol{\rho}([\pi]_Y) = [\boldsymbol{\rho}(\pi)]_Y$. Likewise, if O_Y and O'_Y are κ-equivalent then so are $\hat{\boldsymbol{\rho}}(O_Y)$ and $\hat{\boldsymbol{\rho}}(O'_Y)$ [using the bijection (2)], and $\hat{\boldsymbol{\rho}}$ extends to $\boldsymbol{\rho}^*$ as in (5) by $\boldsymbol{\rho}^*(A) = \hat{\boldsymbol{\rho}}(O_A)$ for any $O_A \in A \in \mathsf{Acyc}(Y)/\!\sim_\kappa$. This map is clearly an involution since $\boldsymbol{\rho}$ itself is an involution. $\square$

Proposition 3 *Let Y be a connected undirected graph. If Y is not bipartite then $\delta(Y) = \frac{1}{2}\kappa(Y)$. If Y is bipartite then $\delta(Y) = \frac{1}{2}(\kappa(Y) + 1)$.*

Proof If Y is not bipartite then Proposition 2 ensures that the involution ρ^* has no fixed points, from which we conclude that $2\delta(Y) = \kappa(Y)$.

For the second statement, we use Proposition 1, Remark 2, and the connectedness of Y to conclude that $\boldsymbol{\rho}^*$ has precisely one fixed point. Since $\boldsymbol{\rho}$ is an involution it follows that $\delta(Y) = \frac{\kappa(Y)-1}{2} + 1 = \frac{\kappa(Y)+1}{2} = \lceil \kappa(Y)/2 \rceil$. □

Thus, we always have $\delta(Y) = \lceil \kappa(Y)/2 \rceil$, and we also have the following characterization of bipartite graphs:

Corollary 1 *A connected graph Y is bipartite if and only if $\kappa(Y)$ is odd.*

For an example where $\boldsymbol{\rho}^*$ has a fixed point see Figure 2 in Example 1. From Proposition 1 we can derive an upper bound for $\kappa(Y)$.

Proposition 4 *If Y is a connected undirected graph on n vertices, then $\kappa(Y) \leq \frac{1}{n}\,\alpha(Y)$.*

Proof By Proposition 1, for any $\pi \in S_Y$, the set $\{O_Y^{\boldsymbol{\sigma}_1(\pi)}, \ldots, O_Y^{\boldsymbol{\sigma}_n(\pi)}\}$ contains n distinct acyclic orientations that are all κ-equivalent and the proof follows. □

This bound is sharp for certain graphs such as the complete graph K_n.

5 Poset Structure of $\boldsymbol{\kappa}$-Equivalence Classes

An edge e of an undirected graph Y is a *bridge* if removing e increases the number of connected components of Y. An edge that is not a bridge is a *cycle-edge*, or equivalently, an edge e is a cycle-edge if it is contained in a cycle traversing e precisely once. The main result in [9] is a recurrence relation for $\kappa(Y)$ under edge deletion and edge contraction.

Theorem 1 ([9]) *Let Y be a finite undirected graph with $e \in \mathrm{e}[Y]$, and let Y'_e be the graph obtained from Y by deleting e, and let Y''_e be the graph obtained from Y by contracting e. Then*

$$\kappa(Y) = \begin{cases} \kappa(Y_1)\kappa(Y_2), & e \textit{ is a bridge linking } Y_1 \textit{ and } Y_2\,, \\ \kappa(Y'_e) + \kappa(Y''_e), & e \textit{ is a cycle-edge}\,. \end{cases} \tag{6}$$

The first case implies that $\kappa(Y)$ is unaffected by removal of bridges, and is relatively straightforward to establish. However, the case where e is a cycle-edge is harder, and was proven in [9] on the level of acyclic orientations. In this section, we will show how one may associate a poset with each κ-equivalence class. The properties of this poset give us better insight into the structure of $\mathsf{Acyc}(Y)/\sim_\kappa$. Additionally, it allows us to construct an alternative proof for Theorem 1.

Throughout, we will let $e = \{v, w\}$ be a fixed cycle-edge of the connected graph Y, and for ease of notation we set $Y' = Y'_e$ and $Y'' = Y''_e$. For $O_Y \in \mathsf{Acyc}(Y)$ we let $O_{Y'}$ and $O_{Y''}$ denote the induced orientations of Y'

and Y''. Notice that $O_{Y'}$ is always acyclic, while $O_{Y''}$ is acyclic if and only if there is no directed path with endpoints v and w in $O_{Y'}$. Finally, we let $[O_Y]$ denote the κ-equivalence class containing O_Y.

The interval $[a,b]$ of a poset $\mathcal{P}$ (where $a \leq b$) is the subposet consisting of all $c \in \mathcal{P}$ such that $a \leq c \leq b$. Viewing a finite poset $\mathcal{P}$ as a directed graph $D_{\mathcal{P}}$, the interval $[a,b]$ contains precisely the vertices that lie on a directed path from a to b, and thus is a vertex-induced subgraph of $D_{\mathcal{P}}$. By assumption, Y contains the edge $\{v,w\}$, so for all $O_Y \in \mathsf{Acyc}(Y)$ either $v \leq_{O_Y} w$ or $w \leq_{O_Y} v$. In this section, we will study the interval $[v,w]$ in the poset O_Y (when $v \leq_{O_Y} w$) and its behavior under clicks.

Definition 1 *Let $\mathsf{Acyc}_{\leq}(Y)$ be the set of acyclic orientations of vertex-induced subgraphs of Y. We define the map*

$$\mathcal{I} \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}_{\leq}(Y) \;,$$

by $\mathcal{I}(O_Y) = [v,w]$ if $v \leq_{O_Y} w$, and by $\mathcal{I}(O_Y) = \varnothing$ (the null graph) otherwise. When $\mathcal{I}(O_Y) \neq \varnothing$ we refer to $\mathcal{I}(O_Y)$ as the vw-interval *of O_Y.*

Elements of $\mathsf{Acyc}_{\leq}(Y)$ can be thought of as subposets of $\mathsf{Acyc}(Y)$. Through a slight abuse of notation, we will at times refer to $\mathcal{I}(O_Y)$ as a poset, a directed graph, or a subset of $\mathrm{v}[O_Y]$. In this last case, it is understood that the relations are inherited from O_Y.

For an undirected path $P = v_1, v_2, \ldots, v_k$ in Y, we define the function

$$\nu_P \colon \mathsf{Acyc}(Y) \longrightarrow \mathbb{Z} \;,$$

where $\nu_P(O_Y)$ is the number of edges oriented as (v_i, v_{i+1}) in O_Y, minus the number of edges oriented as (v_{i+1}, v_i). It is clear that if P is a cycle, then ν_P is preserved under clicks, and in this case, ν_P extends to a map $\nu_P^* \colon \mathsf{Acyc}(Y)/{\sim_\kappa} \longrightarrow \mathbb{Z}$.

We will now prove a series of structural results about the vw-interval. Since $\{v,w\} \in \mathrm{e}[Y]$, every κ-class contains at least one orientation O_Y with $v \leq_{O_Y} w$, and thus there is at least one element O_Y in each κ-class with $\mathcal{I}(O_Y) \neq \varnothing$. The next results shows how we can extend the notion of vw-interval from over $\mathsf{Acyc}(Y)$ to $\mathsf{Acyc}(Y)/{\sim_\kappa}$.

Proposition 5 *The map $\mathcal{I}$ can be extended to a map*

$$\mathcal{I}^* \colon \mathsf{Acyc}(Y)/{\sim_\kappa} \longrightarrow \mathsf{Acyc}_{\leq}(Y) \quad \text{by} \quad \mathcal{I}^*([O_Y]) = \mathcal{I}(O_Y^1) \;,$$

where O_Y^1 is any element of $[O_Y]$ for which $\mathcal{I}(O_Y^1) \neq \varnothing$.

Proof It suffices to prove that $\mathcal{I}^*$ is well-defined. Consider $O_Y^1 \sim_\kappa O_Y^2$ with $v \leq_{O_Y^i} w$ for $i = 1, 2$. To show that $\mathcal{I}(O_Y^1) = \mathcal{I}(O_Y^2)$ let a be a vertex in $\mathcal{I}(O_Y^1)$. Then a lies on a directed path P' from v to w in O_Y^1, say of length $k \geq 2$. Let P be the cycle formed by adding vertex v to the end of P'. Clearly $\nu_P(O_Y^1) = k - 1$ since $O_Y^1(e) = (v,w)$.

By assumption, $O_Y^2 \in [O_Y^1]$ with $v \leq_{O_Y^2} w$. Since ν_P is constant on $[O_Y^1]$ it follows from $\nu_P(O_Y^1) = k-1 = \nu_P(O_Y^2)$ that every edge of P' is oriented identically in O_Y^1 and O_Y^2, and hence that every directed path P' in O_Y^1 is contained in O_Y^2 as well. The reverse inclusion follows by an identical argument. $\square$

In light of Proposition 5, we define the vw-interval of a κ-class $[O_Y]$ to be $\mathcal{I}^*([O_Y])$. The vw-interval will be central in understanding properties of click-sequences. First, we will make a simple observation without proof, which also appears in [16] in the context of admissible sequences in Coxeter theory.

Proposition 6 *Let $O_Y \in \mathsf{Acyc}(Y)$, let $\mathbf{c} = c_1 c_2 \cdots c_m$ be an associated click-sequence, and consider any directed edge (v_1, v_2) in O_Y. Then the occurrences of v_1 and v_2 in $\mathbf{c}$ alternate, with v_1 appearing first.*

Because $\{v, w\} \in \mathrm{e}[Y]$, we can say more about the vertices in $\mathcal{I}(O_Y)$ that appear between successive instances in v and w in a click-sequence.

Proposition 7 *Let $O_Y \in \mathsf{Acyc}(Y)$, and let $\mathbf{c} = c_1 c_2 \cdots c_m$ be an associated click-sequence that contain every vertex of $\mathcal{I}(O_Y)$ at least once and with $c_1 = v$. Then every vertex of $\mathcal{I}(O_Y)$ appears in $\mathbf{c}$ before any vertex in $\mathcal{I}(O_Y)$ appears twice.*

Proof The proof is by contradiction. Assume the statement is false, and let $a \in \mathcal{I}(O_Y)$ be the first vertex whose second instance in $\mathbf{c}$ occurs before the first instance of some other vertex $z \in \mathcal{I}(O_Y)$. If $a \neq v$, then a is not a source in O_Y, and there exists a directed edge (a', a). By Proposition 6, a' must appear in $\mathbf{c}$ before the first instance of a, but also between the two first instances of a. This is impossible, because a was chosen to be the first vertex appearing twice in $\mathbf{c}$. That only leaves $a = v$, and v must appear twice before the first instance of w. However, this contradicts the statement of Proposition 6 because $\{v, w\} \in \mathrm{e}[Y]$. $\square$

The next result shows that for any click-sequence $\mathbf{c}$ that contains every element in $\mathcal{I}(O_Y)$ precisely once, we may assume without loss of generality that the vertices in $\mathcal{I}(O_Y)$ appear consecutively.

Proposition 8 *Let $O_Y \in \mathsf{Acyc}(Y)$ be an acyclic orientation with $v \leq_{O_Y} w$. If $\mathbf{c} = c_1 c_2 \cdots c_m$ is an associated click-sequence containing precisely one instance of w, and no subsequent instances of vertices from $\mathcal{I}(O_Y)$, then there exists a click-sequence $\mathbf{c}' = c_1' c_2' \cdots c_m'$ such that (i) there exists an interval $[p, q]$ of $\mathbb{N}$ with $c_j' \in \mathcal{I}(O_Y)$ iff $p \leq j \leq q$, and (ii) $\mathbf{c}(O_Y) = \mathbf{c}'(O_Y)$.*

Proof We prove the proposition by constructing a desired click-sequence $\mathbf{c}''$ from $\mathbf{c}$ through a series of transpositions where each intermediate click-sequence $\mathbf{c}'$ satisfies $\mathbf{c}(O_Y) = \mathbf{c}'(O_Y)$. Such transpositions are said to have property T.

Let $I = \mathcal{I}(O_Y)$, and let A be the set of vertices in $I^c = \mathrm{v}[Y] \setminus I$ that lie on a directed path in O_Y to a vertex in I (vertices *above* I), and let B be the set of vertices that lie on a directed path in O_Y from a vertex in I (vertices *below* I). Let C be the complement of $I \cup A \cup B$. Two vertices $c_i, c_j \in A \cup B$ with $i < j$ for which there is no element $c_k \in A \cup B$ with $i < k < j$ are said to be *tight*. We will investigate when transpositions of tight vertices in a click-sequence $\mathbf{c}$ of O_Y has property T, and we will see that this is always the case if $c_i \in B$ and $c_j \in A$. Consider the intermediate acyclic orientation after applying successive clicks $c_1 c_2 \cdots c_{i-1}$ to O_Y. Obviously, c_i is a source. At this point, if c_j were not a source, then there would be an adjacent vertex $a \in A$ with the edge $\{a, c_j\}$ oriented (a, c_j). For c_j to be clicked as usual (i.e., as a source), a must be clicked first, but this would break the assumption that c_i and c_j are tight. Therefore, c_i and c_j are both sources at this intermediate step, and so the vertices $c_i, c_{i+1}, \dots, c_j$ are an independent set of sources, and may be permuted in any manner without changing the image of the click sequence. Therefore, the transposition of c_i and c_j in $\mathbf{c}$ has property T, as claimed. By iteratively transposing tight pairs in $\mathbf{c}$, we can construct a click-sequence with the property that every vertex in A comes before every vertex in B. In light of this, we may assume without loss of generality that $\mathbf{c}$ has this property.

The next step is to show that we can move all vertices in A before v, and all vertices in B after w via transpositions having property T. Let a be the first vertex in A appearing after v in the click sequence $\mathbf{c}$. We claim that the transposition moving a to the position directly preceding v has property T. This is immediate from the observation that when v is to be clicked, a is a source as well, by the definition of A, thus it may be clicked before v, without preventing subsequent clicks of vertices up until the original position of a. Therefore, we may one-by-one move the vertices in A that are between v and w, in front of v. An analogous argument shows that we may move the vertices in B that appear before w to a position directly following w. In the resulting click-sequence $\mathbf{c}'$, the only vertices between v and w are either in I or C. The subgraph of the directed graph O_Y induced by C is a disjoint union of weakly connected components, and none of the vertices are adjacent to I. By definition of A and B, there cannot exist a directed edge (c, a) or (b, c), where $a \in A$, $b \in B$, and $c \in C$. Thus for each weakly connected component of C, the vertices in the component can be moved within $\mathbf{c}'$, preserving their relative order, to a position either (i) directly after the vertices in A and before v, or (ii) directly after w and before the vertices of B. Call this resulting click-sequence $\mathbf{c}''$. As we just argued, all the transpositions occurring in the rearrangement $\mathbf{c} \mapsto \mathbf{c}''$ has property T, and $\mathbf{c}''$ contains all of the vertices in I in consecutive order, and this proves the result. $\square$

We remark that the last two results together imply that for the interval $[p, q]$ in the statement of Proposition 8, $c_p = v$, $c_q = w$, and the sequence $c_p c_{p+1} \cdots c_q$ contains every vertex in $\mathcal{I}(O_Y)$ precisely once. A simple induction argument implies the following.

Corollary 2 *Suppose that $O_Y \in \mathsf{Acyc}(Y)$ with $v \leq_{O_Y} w$, and let $\mathbf{c} = c_1c_2\cdots c_m$ be a click-sequence where w appears k times. Then there exists a click-sequence $\mathbf{c}' = c_1'c_2'\cdots c_m'$ such that (i) there are k disjoint intervals $[p_i, q_i]$ of $\mathbb{N}$ such that $c_j \in \mathcal{I}(O_Y)$ iff $p_i \leq j \leq q_i$ for some i, and (ii) $\mathbf{c}(O_Y) = \mathbf{c}'(O_Y)$.*

Proof The argument is by induction on k. When $k = 1$, the statement is simply Proposition 8. Suppose the statement holds for all $k \leq N$, for some $N \in \mathbb{N}$, and let $\mathbf{c}$ be a click-sequence containing $N+1$ instances of w. Let c_ℓ be the second instance of v in $\mathbf{c}$, and consider the two click-sequences $\mathbf{c}_i := c_1c_2\cdots c_{\ell-1}$ and $\mathbf{c}_f := c_\ell c_{\ell+1}\cdots c_m$. By Proposition 8, there exists an interval $[p_1, q_1]$ with $p_1 < q_1 < \ell$, and by the induction hypothesis, there exists k intervals $[p_2, q_2], \ldots, [p_{k+1}, q_{k+1}]$ with $\ell \leq p_2 < q_2 < \cdots < p_{k+1} < q_{k+1}$ such that if $c_j \in \mathcal{I}(O_Y)$, then $p_i \leq j \leq q_i$ for some $i = 1, \ldots, k+1$. $\square$

Let $\eta_e \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}(Y')$ be the restriction map that sends O_Y to $O_{Y'}$. Clearly, this map extends to a map $\eta_e^* \colon \mathsf{Acyc}(Y)/\!\sim_\kappa \longrightarrow \mathsf{Acyc}(Y')/\!\sim_\kappa$. Define

$$\mathcal{I}_e^* \colon \mathsf{Acyc}(Y')/\!\sim_\kappa \longrightarrow \mathsf{Acyc}_{\leq}(Y)$$

by $\mathcal{I}_e^*([O_{Y'}]) = \mathcal{I}(O_Y^1)$ for any $O_Y^1 \in [O_Y]$ such that $\eta_e^*([O_Y]) = [O_{Y'}]$ with $|\mathcal{I}(O_Y^1)| \geq 3$, and $\mathcal{I}_e^*([O_{Y'}]) = \{v, w\}$ if no such acyclic orientation O_Y^1 exists.

Proposition 9 *The map $\mathcal{I}_e^*$ is well-defined, and the diagram*

$$\begin{tikzcd}
\mathsf{Acyc}(Y)/\!\sim_\kappa \arrow[r, "\mathcal{I}^*"] \arrow[d, "\eta_e^*"'] & \mathsf{Acyc}_{\leq}(Y) \\
\mathsf{Acyc}(Y')/\!\sim_\kappa \arrow[ur, dashed, "\mathcal{I}_e^*"'] &
\end{tikzcd}$$

commutes.

Proof Let $[O_{Y'}] \in \mathsf{Acyc}(Y')/\!\sim_\kappa$. If there is at most one orientation $O_Y \in \mathsf{Acyc}(Y)$ such that $|\mathcal{I}(O_Y)| \geq 3$ and $\eta_e(O_Y) \in [O_{Y'}]$, or if all orientations of the form O_Y^1 in the definition of $\mathcal{I}_e^*$ are κ-equivalent, then both statements of the proposition are clear. Assume therefore that there are acyclic orientations $O_Y^\pi, O_Y^\sigma \in \mathsf{Acyc}(Y)$ with $O_Y^\pi \nsim_\kappa O_Y^\sigma$, but $\eta_e^*([O_Y^\pi]) = \eta_e^*([O_Y^\sigma])$ and $|\mathcal{I}(O_Y^\pi)|, |\mathcal{I}(O_Y^\sigma)| \geq 3$. It suffices to prove that in this case,

$$\mathcal{I}(O_Y^\pi) = \mathcal{I}(O_Y^\sigma)\ . \tag{7}$$

This is equivalent to showing that the set of vw-paths (directed paths from v to w) in $O_{Y'}^\pi$ is the same as the set of vw-paths in $O_{Y'}^\sigma$. From this it will also follow that the diagram commutes. By assumption, both of these orientations contain at least one vw-path. We will consider separately the cases when these orientations share or do not share a common vw-path.

Case 1: $O_{Y'}^\pi$ and $O_{Y'}^\sigma$ share no common vw-path. Let P_1 be a length-k_1 vw-path in $O_{Y'}^\pi$, and let P_2 be a length-k_2 vw-path in $O_{Y'}^\sigma$. Suppose that in $O^\pi_{Y'}$ there are k_2^+ edges along P_2 oriented from v to w, and k_2^- edges oriented from w to v. Likewise, suppose that in $O^\sigma_{Y'}$ there are k_1^+ edges along P_1 oriented from v to w, and k_1^- edges oriented from w to v. If $C = P_1P_2^{-1}$ (the cycle formed by traversing P_1 followed by P_2 in reverse), then

$$\nu_C(O^\pi_{Y'}) = k_1^+ + k_1^- + k_2^- - k_2^+, \qquad \nu_C(O^\sigma_{Y'}) = k_1^+ - k_1^- - k_2^- - k_2^+ \ .$$

Equating these values yields $k_1^- + k_2^- = 0$, and since these are non-negative integers, $k_1^- = k_2^- = 0$. We conclude that P_1 is a vw-path in $O^\sigma_{Y'}$ and P_2 is a vw-path in $O^\pi_{Y'}$, contradicting the assumption that $O^\pi_{Y'}$ and $O^\sigma_{Y'}$ share no common vw-paths.

Case 2: $O^\pi_{Y'}$ and $O^\sigma_{Y'}$ share a common vw-path P_1, say of length k_1. If these are the only vw-paths, we are done. Otherwise, assume without loss of generality that P_2 is another vw-path in $O^\pi_{Y'}$, say of length k_2. Then if $C = P_1P_2^{-1}$, we have $\nu_C(O^\pi_{Y'}) = k_1 - k_2$, and hence $\nu_C(O^\sigma_{Y'}) = k_1 - k_2$. Therefore, P_2 is a vw-path in $O^\sigma_{Y'}$ as well. Because P_2 was arbitrary, we conclude that $O^\pi_{Y'}$ and $O^\sigma_{Y'}$ share the same set of vw-paths. Since Case 1 is impossible, we have established (7), and the proof is complete. □

Let $O_Y \in \mathsf{Acyc}(Y)$ and assume $I = \mathcal{I}(O_Y)$ has at least two vertices. We write Y_I for the graph formed from Y by contracting all vertices in I to a single vertex denoted V_I. If I only contains v and w then $Y_I = Y_e''$. Moreover, O_Y gives rise to an orientation O_{Y_I} of Y_I, and this orientation is clearly acyclic.

Proposition 10 *Let $O^1_Y, O^2_Y \in \mathsf{Acyc}(Y)$ and assume $\mathcal{I}(O^1_Y) = \mathcal{I}(O^2_Y)$. If $O^1_Y \nsim_\kappa O^2_Y$ then $[O^1_{Y_I}] \nsim_\kappa [O^2_{Y_I}]$.*

Proof We prove the contrapositive statement. Set $I = \mathcal{I}(O^1_Y)$, suppose $|I| = k$, and let $v_1v_2\cdots v_k$ be a linear extension of I. For any click-sequence $\mathbf{c}_I$ between two acyclic orientations $O^1_{Y_I}$ and $O^2_{Y_I}$ in $\mathsf{Acyc}(Y_I)$, let $\mathbf{c}$ be the click-sequence formed by replacing every occurrence of $c_i = V_I$ in $\mathbf{c}_I$ by the sequence $v_1\cdots v_k$. Then $\mathbf{c}(O^1_Y) = O^2_Y$ and $O^1_Y \sim_\kappa O^2_Y$ as claimed. □

6 Proof of Theorem 1

In this section, we will utilize the results in the previous section to establish a bijection from $\mathsf{Acyc}(Y)/\!\sim_\kappa$ to $\big(\mathsf{Acyc}(Y_e')/\!\sim_\kappa \cup \mathsf{Acyc}(Y_e'')/\!\sim_\kappa\big)$ for any cycle-edge e, which will in turn imply Theorem 1.

For $[O_Y] \in \mathsf{Acyc}(Y)/\!\sim_\kappa$, let O^π_Y denote an orientation in $[O_Y]$ such that $\pi = v\pi_2\cdots\pi_n$ and $w = \pi_i$ for i minimal. Define the map

$$\Theta\colon \mathsf{Acyc}(Y)/\!\sim_\kappa \longrightarrow \big(\mathsf{Acyc}(Y')/\!\sim_\kappa\big) \bigcup \big(\mathsf{Acyc}(Y'')/\!\sim_\kappa\big) \tag{8}$$

by

$$[O_Y] \overset{\Theta}{\longmapsto} \begin{cases} [O^\pi_{Y''}], & \exists O^\pi_Y \in [O_Y] \text{ with } \pi = vw\pi_3\cdots\pi_n \\ [O^\pi_{Y'}], & \text{otherwise.} \end{cases} \tag{9}$$

Note that $[O_Y]$ is mapped into $\mathsf{Acyc}(Y'')/\sim_\kappa$ if and only if the only vertices in $\mathcal{I}^*_e([O_Y])$ are v and w. Since κ-equivalence over Y implies κ-equivalence over Y', Θ does not depend on the choice of π, and is thus well-defined. The results we have derived for the vw-interval now allow us to establish the following:

Theorem 2 *The map Θ is a bijection.*

Proof We first prove that Θ is surjective. Let $I = \{v, w\}$ and consider an element $[O_{Y''}] \in \mathsf{Acyc}(Y'')/\sim_\kappa$ with $O^\pi_{Y''} \in [O_{Y''}]$ where $\pi = V_I\pi_2 \cdots \pi_{n-1}$. Let $\pi^+ = vw\pi_2 \cdots \pi_{n-1} \in S_Y$. Clearly $[O^{\pi^+}_Y] \in \mathsf{Acyc}(Y)/\sim_\kappa$ is mapped to $[O_{Y''}]$ by Θ.

Next, consider an element $[O_{Y'}] \in \mathsf{Acyc}(Y')/\sim_\kappa$. If there is no element $O^\pi_{Y'}$ of $[O_{Y'}]$ such that $\pi = vw\pi_3 \cdots \pi_n$, then no elements of $[O_Y]$ are of this form either, and by definition $[O_{Y'}]$ has a preimage under Θ. We are left with the case where $[O_{Y'}]$ contains an element $O^\pi_{Y'}$ such that $\pi = vw\pi_3 \cdots \pi_n$, and we must show that there exists $O^{\pi'}_{Y'} \in [O_{Y'}]$ such that $[O^{\pi'}_Y]$ contains no element of the form O^σ_Y with $\sigma = vw\sigma_3 \cdots \sigma_n$. Note that if $\sigma = vw\sigma_3 \cdots \sigma_n$, then the vertices in $\mathcal{I}(O^\sigma_Y)$ are precisely v and w. If the orientation $O_{Y'}$ had a directed path from v to w, then the corresponding orientation $O_Y \in \mathsf{Acyc}(Y)$ formed by adding the edge e with orientation (v, w) has vw-interval of size at least 3, so by Proposition 5, the acyclic orientation O_Y cannot be κ-equivalent to any orientation O^σ_Y such that $\sigma = vw\sigma_3 \cdots \sigma_n$.

Thus it remains to consider the case when $[O_{Y'}]$ contains no acyclic orientation with a directed path from v to w. Pick any simple undirected path P' from v to w in Y', which exists because e is a cycle-edge. Choose an orientation in $[O_{Y'}]$ for which $\nu_{P'}$ is maximal. Without loss of generality we may assume that $O_{Y'}$ is this orientation. Let $O_Y \in \mathsf{Acyc}(Y)$ be the orientation that agrees with $O_{Y'}$, and with e oriented as (w, v). Since we have assumed that there is no directed path from v to w this orientation is acyclic. We claim that for any $\sigma = vw\sigma_3 \cdots \sigma_n$ one has $O^\sigma_Y \notin [O_Y]$. To see this, assume the statement is false. Let P be the undirected cycle in Y formed by adding the edge e to the path P'. Because e is oriented as (v, w) in O^σ_Y and as (w, v) in O_Y, we have $\nu_P(O^\sigma_Y) = \nu_{P'}(O^\sigma_{Y'}) - 1$ and $\nu_P(O_Y) = \nu_{P'}(O_{Y'}) + 1$. If O_Y and O^σ_Y were κ-equivalent, then

$$\nu_{P'}(O^\sigma_{Y'}) - 1 = \nu_P(O^\sigma_Y) = \nu_P(O_Y) = \nu_{P'}(O_Y) + 1 \ ,$$

and thus $\nu_{P'}(O^\sigma_{Y'}) = \nu_{P'}(O_Y) + 2$. Any click sequence mapping O_Y to O^σ_Y is a click-sequence from $O_{Y'}$ to $O^\sigma_{Y'}$. Therefore, $O^\sigma_{Y'} \in [O_{Y'}]$, which contradicts the maximality of $\nu_{P'}(O_{Y'})$. We therefore conclude that $O^\sigma_Y \notin [O_Y]$, that $\Theta([O_Y]) = [O_{Y'}]$, and hence that Θ is surjective.

We next prove that Θ is an injection. By Proposition 10 (with $I = \{v, w\}$), Θ is injective when restricted to the preimage of $[O_{Y''}]$ under Θ. Thus it suffices to show that every element in $\mathsf{Acyc}(Y')/\sim_\kappa$ has a unique preimage under Θ. By Proposition 9, every preimage of $[O_{Y'}]$ must have the same vw-interval I, containing $k > 2$ vertices. Suppose there were preimages

$[O^\pi_Y] \neq [O^\sigma_Y]$ of $[O_{Y'}]$. By Proposition 10, it follows that $O^\pi_{Y_I} \nsim_\kappa O^\sigma_{Y_I}$. We will now show that this leads to a contradiction.

Assume that $\mathbf{c} = c_1 \cdots c_m$ is a click-sequence from $O^\pi_{Y'}$ to $O^\sigma_{Y'}$. If one of π or σ is not κ-equivalent to a permutation with vertices v and w in succession, then their corresponding κ-classes would be unchanged by the removal of edge e. In light of this, we may assume that $\pi = v\pi_2 \dots \pi_{n-1}w$ and $\sigma = v\sigma_2 \dots \sigma_{n-1}w$, and thus that $c_1 = v$ and $c_m = w$. By Proposition 8, we may assume that the vertices in I appear in $\mathbf{c}$ in some number of disjoint consecutive "blocks," i.e., subsequences of the form $c_i \cdots c_{i+k-1}$. Replacing each of these blocks with V_I yields a click-sequence from $O^\pi_{Y_I}$ to $O^\sigma_{Y_I}$, contradicting the fact that $O^\pi_{Y_I} \nsim_\kappa O^\sigma_{Y_I}$. Therefore, no such click sequence $\mathbf{c}$ exists, and Θ must be an injection, and the proof is complete. □

Clearly, Theorem 2 implies Theorem 1. It is also interesting to note that the bijection $\beta_e \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}(Y'_e) \cup \mathsf{Acyc}(Y''_e)$ in (3) does *not* extend to a map on κ-classes.

7 Discrete Dynamical Systems, Coxeter Groups, Node-firing Games, and Quiver Representations

We conclude with a brief discussion of how the equivalence relations studied in this paper arise in various areas of mathematics. The original motivation came from the authors' interest in *sequential dynamical systems* (SDSs). The equivalence relation $\sim_Y$ arises naturally in the study of functional equivalence of these systems. This can be seen as follows. Given a graph Y with vertex set $\{1, 2, \dots, n\}$ as above, a state $x_v \in K$ is assigned to each vertex v of Y for some finite set K. The *system state* is the tuple consisting of all the vertex states, and is denoted $x = (x_1, \dots, x_n) \in K^n$. The sequence of states associated to the 1-neighborhood $B_1(v;Y)$ of v in Y (in some fixed order) is denoted $x[v]$. A sequence of vertex functions $(f_i)_i$ with $f_i \colon K^{d(i)+1} \longrightarrow K$ induces Y-local functions $F_i \colon K^n \longrightarrow K^n$ of the form

$$F_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, f_i(x[i]), x_{i+1}, \dots, x_n) \ .$$

The sequential dynamical system map with update order $\pi = (\pi_i)_i \in S_Y$ is the function composition

$$[\mathfrak{F}_Y, \pi] = F_{\pi(n)} \circ F_{\pi(n-1)} \circ \cdots \circ F_{\pi(2)} \circ F_{\pi(1)} \ . \tag{10}$$

By construction, if $\pi \sim_Y \pi'$ holds then $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \pi']$ are identical as functions. Thus, $\alpha(Y)$ is an upper bound for the number of functionally non-equivalent SDS maps that can be generated over the graph Y for a fixed sequence of vertex functions. For any graph Y, there exist Y-local functions for which this bound is sharp [11]. A weaker form of equivalence is *cycle equivalence*, which means that the dynamical system maps are conjugate (using the discrete topology) when restricted to their sets of periodic points.

In the language of graph theory, this means their periodic orbits are isomorphic as directed graphs. The following result shows how κ- and δ-equivalent update orders yield dynamical system maps that are cycle equivalent.

Theorem 3 *For any finite set K of vertex states, and for any $\pi \in S_Y$, the* SDS *maps $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \boldsymbol{\sigma}_s(\pi)]$ are cycle equivalent. For vertex states $K = \mathbb{F}_2$ the* SDS *maps $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \boldsymbol{\rho}(\pi)]$ are cycle equivalent as well.*

We refer to [8] for the proof of this result. Two sequential dynamical systems are cycle equivalent if their phase space digraphs are isomorphic when restricted to the cycles. The paper [8] contains additional background on equivalences of sequential dynamical systems as well as applications of κ-equivalence to the structural properties of their phase spaces.

There is also a connection between κ-equivalence and the theory of Coxeter groups. There is a natural bijection between the set of Coxeter elements $\prod_i s_{\pi(i)}$ of a Coxeter group (see, e.g. [1]) with generators s_i for $1 \leq i \leq n$ and Coxeter graph Y (ignoring order labels), and the set of equivalence classes $[\pi]_Y$. This is clear since the commuting generators are precisely those that are not connected in Y. Let $C(W, S)$ denote the set of Coxeter elements of a Coxeter group W with generating set $S = \{s_1, \ldots, s_n\}$. It is shown in [14] that there is a bijection

$$C(W, S) \longrightarrow \mathsf{Acyc}(Y)\ . \tag{11}$$

Moreover, conjugating a Coxeter element $\prod s_{\pi(i)}$ by $s_{\pi(1)}$ corresponds to a cyclic shift, i.e.,

$$s_{\pi(1)}(s_{\pi(1)} s_{\pi(2)} \cdots s_{\pi(n)}) s_{\pi(1)} = s_{\pi(2)} \cdots s_{\pi(n)} s_{\pi(1)}\ , \tag{12}$$

since each generator s_i is an involution. Thus $\kappa(Y)$ is an upper bound for the number of conjugacy classes of Coxeter elements of a Coxeter group that has (unlabeled) Coxeter graph Y. This bound is known to be sharp in certain cases [15], but sharpness in the general case is still an open question.

Click-sequences are closely related to c-admissible sequences of a Coxeter element c. Recently, the structure of these sequences was studied and used to prove that a power of a Coxeter element of an infinite group is reduced [16].

The *chip-firing* game was introduced by Björner, Lovász, and Shor [2]. It is played over an undirected graph Y, and each vertex is given some number (possibly zero) of chips. If vertex i has degree d_i, and at least d_i chips, then a legal move (or a "click") of vertex i is a transfer of one chip to each neighboring vertex. This may be viewed as a generalization of a source-to-sink move for acyclic orientations where the out-degree of a vertex plays the role of the chip count. The chip-firing game is closely related to the *numbers game* [1]. In the numbers game over a graph Y, the legal sequences of moves are in 1–1 correspondence with the reduced words of the Coxeter group with Coxeter graph Y. For an excellent summary and comparison of these games, see [4].

A quiver is a finite directed graph (loops and multiple edges are allowed), and appears primarily in the study of representation theory. A quiver Q with a field K gives rise to a path algebra KQ, and there is a natural correspondence between KQ-modules, and K-representations of Q. In fact, there is an equivalence between the categories of quiver representations, and modules over path algebras. A path algebra is finite-dimensional if and only if the quiver is acyclic, and the modules over finite-dimensional path algebras form a reflective subcategory. A *reflection functor* maps representations of a quiver Q to representations of a quiver Q', where Q' differs from Q by a source-to-sink operation [10]. We note that while the composition of n source-to-sink operations (one for each vertex) maps a quiver back to itself, the corresponding composition of reflection functors is not the identity, but rather a *Coxeter functor*. In fact, the same result in [16] about powers of Coxeter elements being reduced was proven previously using techniques from the representation theory of quivers [6].

We conclude with a remark on the evaluation of the functions $\alpha(Y)$ and $\kappa(Y)$. They both satisfy recurrences under edge deletion and contraction, and may be computed through evaluations of the Tutte polynomial T_Y of Y. It is well-known that $\alpha(Y) = T_Y(2,0)$, and in [9], we showed that $\kappa(Y) = T_Y(1,0)$. Other quantities counted by $T_Y(1,0)$ include the number of acyclic orientations of Y with a unique sink at a fixed vertex [5], and the Möbius invariant of the intersection lattice of the graphic hyperplane arrangement of Y [12]. Some of the results in this paper have a natural interpretation in the language of the Tutte polynomial. For example, Corollary 1 tells us that a connected graph Y is bipartite if and only if $T_Y(1,0)$ is odd. Proposition 4 implies that $n \cdot T_Y(1,0) \leq T_Y(2,0)$. We also point out a previous study of the Tutte polynomial in the context of the chip-firing game [7]. We hope this paper will motivate further explorations of the connections between these topics, as also provide insights useful to some of the open problems in Coxeter theory, in particular the sharpness of the bound $\kappa(Y)$ for the enumeration of conjugacy classes of Coxeter elements.

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